Let $o_i(x)$ be odd
& $e_i(x)$ be even
then:

① $e_i(x)+e_j(x)$, $e_i(x)\cdot e_j(x)$: even
② $o_i(x)+o_j(x)$ odd ; $o_i(x)\cdot o_j(x)$ even
③ $o_i(x)+e_j(x)$ neither even nor odd ; $o_i(x)\cdot e_j(x)$ odd

4/7) <u>Cosine Series</u>

Suppose f, f' are piecewise continuous on $[-L, L]$ and f is an even periodic function of period $2L$. We would like to calculate its Fourier series.

<u>Recall</u>: Let f be an odd function and g an even function. Then:

$$\int_{-L}^{L} f(x)\,dx = 0 \quad \& \quad \int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

Notice that $f(x)\cos\frac{n\pi x}{L}$ is even and $f(x)\sin\frac{n\pi x}{L}$ is odd.

To calculate the Fourier series of f i.e. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ we need to calculate the coefficients.

<u>Recall</u> $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ & $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,d$

<u>Calculate:</u>

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \forall n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{\text{odd}}\,dx = 0 \quad \forall n \geq 1$$

So an even periodic function has Fourier expansion: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$.

<u>Sine Series</u>: Suppose f, f' are p.w. cts. on $[L, L]$ and that f is odd and periodic with period $2L$.

Notice now that:

① $f(x) \cos \frac{n\pi x}{L}$ is odd

② $f(x) \sin \frac{n\pi x}{L}$ is even

so that:

$$① \Rightarrow a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = 0 \quad \forall n \geq 0$$

$$② \Rightarrow b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

Thus our Fourier expansion of f will be:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

<u>Periodic Extentions of Functions</u>

Suppose f is some function defined on $[0, L]$. We could extend this function to a periodic function on $\mathbb{R}$ in several way, for example we could create $\tilde{f}$ (the extention) by $\tilde{f}(x) = \begin{cases} f(x), & 0 \leq x \leq L \\ f(L), & -L \leq x < 0 \end{cases}$ and declare $f(x+2L) = f(x)$.

However there are some rather nice extentio
which make the Fourier series simpler..
Let's focus on some specific extentions.

Even Extention

g has period $2L$ where

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0. \end{cases}$$

This yeilds a cosine Fourier series.

Odd Extention

h has period $2L$ where

$$h(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, L \\ -f(-x), & -L < x < 0. \end{cases}$$

This yeilds a sine Fourier series.

Now let's see an example:

Ex 1:

10.4

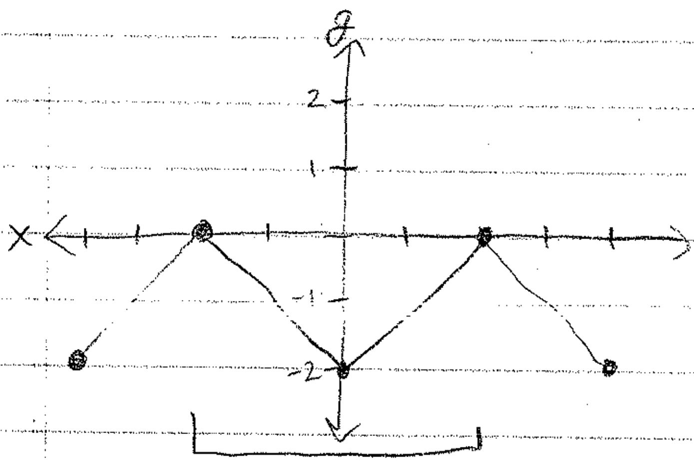
9 $f(x) = 2 - x, \quad 0 < x < 2$

Find the even and odd extentions of f and their Fourier series.

Sol: Note $L=2$

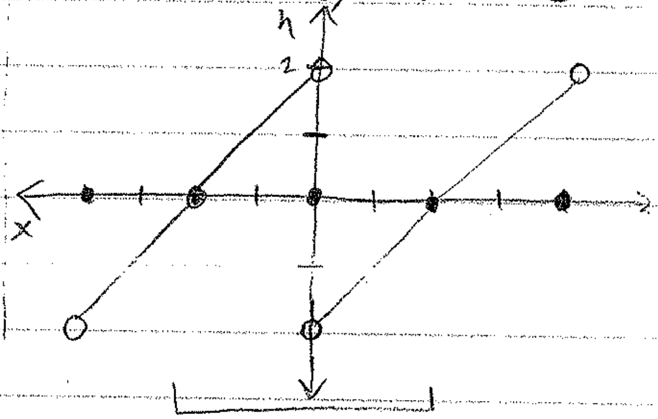
Even Extention

$$g(x) = \begin{cases} x-2, & 0 \le x \le 2 \\ -(x+2), & -2 < x < 0 \end{cases}$$



Odd Extention

$$h(x) = \begin{cases} x-2, & 0<x<2 \\ 0, & x=0,2 \\ x+2, & -2<x<0 \end{cases}$$



For g:

$$a_0 = \frac{1}{2}\int_{-2}^{2} g(x)\,dx = \int_0^2 g(x)\,dx = \int_0^2 x-2\,dx$$

$$= \left(\tfrac{1}{2}x^2 - 2x\right)\Big|_0^2 = 2-4 = -2$$

$x-2$ $du=dx$

$\cos\frac{n\pi x}{2}\,dx$

$\frac{2}{n\pi}\sin\frac{n\pi x}{2}$

$$a_n = \frac{1}{2}\int_{-2}^{2} g(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 (x-2)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2x-4}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}\,dx$$

$$= 0-0+\left[\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ even} \\ \frac{-8}{n^2\pi^2}, & n \text{ odd} \end{cases}$$

$$\Rightarrow g(x) = -1 + \sum_{n\text{ odd}} \frac{-8}{n^2\pi^2}\cos\frac{n\pi x}{2} = -1 + \sum_{m=1}^{\infty} \frac{-8}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi x}{2}$$

$u = x-2 \quad du = dx$
$dv = \sin\frac{n\pi x}{2}\,dx$
$v = \frac{-2}{n\pi}\cos\frac{n\pi x}{2}$

<u>For h:</u>

$$b_n = \frac{1}{2}\int_{-2}^{2} h(x)\sin\frac{n\pi x}{2}\,dx = \int_0^2 (x-2)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{-2x+4}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx$$

$$= 0 - \left(\frac{4}{n\pi}\right) + 0 = \frac{-4}{n\pi}$$

So: $$h(x) = \sum_{n=1}^{\infty} \frac{-4}{n\pi}\sin\frac{n\pi x}{2} = \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

The Wave Equation

Separation of Variables

Given the wave equation w/ BCs & ICs

$$(WE)\quad \begin{cases} ① \; a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \; t > 0 \\ ② \; u(0,t) = u(L,t) = 0, \quad t \geq 0 \quad \text{(fixed endpoints)} \\ ③ \; u(x,0) = f(x) \;\text{(initial position)}, \; u_t(x,0) = 0 \;\text{(ini. vel.)} \quad 0 \leq x \leq L \end{cases}$$

solve for u.

Suppose $u(x,t) = X(x)\,T(t)$.
Then plug into ①:

$$a^2 X''(x)T(t) = X(x)T''(t)$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda \quad ④$$

(4) Separates into:

$$\begin{cases} X''(x) + \lambda X(x) = 0 & (5) \\ T''(t) + a^2\lambda T(t) = 0 & (6) \end{cases}$$

using (2) with (5) we get:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \quad (7)$$

(3) + (6) gives:

$$\begin{cases} T''(t) + a^2\lambda T(t) = 0 \\ T'(0) = 0 \end{cases} \quad (8)$$

Solve (7): 3 cases

$\lambda = \mu^2$ (assume $\mu > 0$)

$X'' + \mu^2 X = 0 \Rightarrow X(x) = c_1 \cos \mu x + c_2 \sin \mu x$

$X(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$X(L) = 0 \Rightarrow c_2 \sin \mu L = 0 \Rightarrow \sin \mu L = 0$

$\Rightarrow \mu L = n\pi \Rightarrow \mu = \frac{n\pi}{L} \Rightarrow \underbrace{\lambda_n}_{\text{e.vals}} = \frac{n^2\pi^2}{L^2} \quad n \in \mathbb{N}$

So the eigenfunctions are $X_n(x) = \sin \frac{n\pi x}{L}, \ n \in \mathbb{N}$

$\lambda = -\mu^2$ has no solution

$\lambda = 0$ has no nontrivial solution

Now solve ⑧:

$$T'' + \lambda a^2 T = 0 \Rightarrow T'' + \frac{n^2\pi^2 a^2}{L^2} T = 0$$

$$\Rightarrow T = k_1 \cos \frac{n\pi a t}{L} + k_2 \sin \frac{n\pi a t}{L}$$

$$T' = -\frac{n\pi a}{L} k_1 \sin \frac{n\pi a t}{L} + \frac{n\pi a}{L} k_2 \cos \frac{n\pi a t}{L}$$

$\Rightarrow k_2 = 0$

So $T_n(t) = \cos \frac{n\pi a t}{L}$.

Now $u_n(x,t) = X_n(x) T_n(t)$

$$= \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}, \quad n \in \mathbb{N}$$

Suppose $u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$.

Now the remaining IC $u_t(x,0) = f(x)$ gives

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x)$$

hence c_n are the coef. of the sine series of f: $c_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n \in \mathbb{N}$

So the solution to (WE) is:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L} \quad \text{with } c_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$